Second topological conjugate transformation in symbolic dynamics

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A topological conjugate transformation defined as the joint actions of both the Derrida-Gervois-Pomeau (DGP) $*$ product operation $QC*$ in the symbolic space (or its corresponding parameter space) and the mapping $f^{|QC|}$ in the symbolic dynamics of the interval, which with respect to the first topological conjugate transformation (the merely action of $QC*$) is called the second topological conjugate transformation, is found. It reveals conspicuously clustering of the orbital points and preserves the topological entropy of the dynamical systems. In analogy to the first topological conjugate transformation, there exist also infinitely many second topological conjugate maps. The second topological conjugate transformation provides a topological foundation for Feigenbaum's [J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979)] universalities and a basic topological method for discriminating the compound words in the sense of the DGP $*$ product in the symbolic space Σ_2 of two letters. Therefore it opens up a way to seek the generalized $*$ product for the more complex dynamical systems. [S1063-651X(98)06405-8]

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I. INTRODUCTION

Simple one-dimensional iterative systems display a rich connotation [1]. Many studies reveal that the Derrida-Gervois-Pomeau (DGP) $*$ product [2] plays a key role in the understanding of the regularities in the chaotic phenomena aroused by nonlinearities [3]. It explains the self-similarity and self-embedding phenomena therein and provides a rigorous symbolic formalism for expounding the invariance of dynamical systems. Therefore, further research on the characteristics of the DGP product $QC*$ becomes increasingly important in exploring the regularities of nonlinear complex phenomena. First of all, $QC*$ allows us to describe the relation between two kneading invariants of two different maps related by the renormalization transformation [4]. Second, $QC*$ is a symbolic representation of Feigenbaum period-p-tupling bifurcation processes [i.e., the kneading sequence series $(QC)*^n$, $n=1,2,...$] and the universal constants $\delta(QC)$ and $\alpha(QC)$ characterize the contraction and self-embedding in the parametric space under the action of the operator $QC*$, while the renormalization group can be comprehended as a result of continuously acting $QC*$ in the symbolic space or its corresponding parametric space of the dynamical systems. Third, recent results [3] show that for any superstable kneading sequence QC, the subinterval $QC*[L^\infty,RL^\infty]$ corresponds to the entire interval $[L^\infty,RL^\infty]$ under a one-to-one mapping $QC*$ and it is a $1/\delta(Q)$ times contraction of the entire interval; if $QC\neq(RC)*^n$, then the subinterval $QC*[L^\infty,RL^\infty]$ forms an equal topological entropy class with a topological entropy $h_t(f_{\lambda_{QC}}(x))$ that expresses a step in the entropy devil's staircase. Because the action of the operator $QC*$ keeps the topological entropy constant, we call the operation $QC*$: $\Sigma_2\to\Sigma_2$ in the symbolic space *the first topological conjugate transformation* from the viewpoint of the parameter shift $h_t(f_{\lambda_{QC*S}}(x))=h_t(f_{\lambda_{QC}}(x))$ for $QC\neq(RC)*^n$, $S\in\Sigma_2$, where Σ_2 is the set formed by all the admissible words of two letters [3]. It is important that the main $[(QC)*^n]$ and associated $[(QC)*^n*S]$ Feigenbaum universalities are confined within the equal topological entropy class. Accordingly, the operator $QC*$ reveals the vigorous global regularities and the order structures in the chaotic phenomena. In this way, the DGP $*$ product is a fundamental tool to study the topological and metric behaviors in the unimodal maps and other piecewise monotone maps [5,6].

The mapping f is a basic method to study the behavior of a dynamical system. For a period-p system, the mapping can show its periodicity by p times iterations: $f^p(x)=x$. For an arbitrary orbit of a dynamical system represented by a symbolic sequence (word) S with parameter λ_S, on the one hand, the coordinates (or positions) of its orbital points in the symbolic space are labeled by the shift operations $\{\varphi^kS\}_{k=0}^{k=|S|}$ (here $|S|$ denotes the length of word S) according to the Metropolis-Stein-Stein (MSS) order [7]; on the other hand, the transition of its orbital points are successively connected by the shift mapping φ according to the order of iteration times k. Hence these two aspects describe a *symbolic orbit* or *transition pattern* in the symbolic space. When acting with the operator $QC*$, we obtain a new orbit in the symbolic space determined by the parameter λ_{QC*S}. This new orbit has the topological entropy $h_t(f_{\lambda_{QC}}(x))$ in terms of mapping f^1. However, if inspecting the new orbit in terms of $|QC|$ times iteration $f^{|QC|}$, then we observe an interesting phenomenon, namely, there exists an invariant of the topological entropy $h_t(f^{|QC|}_{\lambda_{QC*S}}(x))=h_t(f^1_{\lambda_S}(x))$ under the joint actions of $QC*$ and $f^{|QC|}$. Thus we can define the joint actions of the operator $QC*$ in the symbolic or parameter space and the $|QC|$ times mapping of f (i.e., $f^{|QC|}$) in the interval dy-

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namics as the *second topological conjugate transformation*. Like the first transformation, this second one establishes another invariant that also preserves the topological entropy as constant. These two kinds of transformations are different: The first one keeps the topological entropy of the left superstable periodic word QC in the product $QC*S$, while the second one keeps that of the right word S.

One of the aims in this paper is to establish the second conjugate transformation. We first find that there are $|QC|$ clusters of orbital points in the symbolic space exerted by the operator $QC*$. Each cluster is self-closed under the mapping $f^{|QC|}$ and forms an invariant subinterval whose transition pattern is consistent with or opposite to that of the original orbit under mapping f^1. Moreover, this leads to the block diagonalization of the Stefan transition matrix of compound mapping $f^{|QC|}$ and every block is either the same as or the transpose (rotation of 180°) of the Stefan transition matrix of mapping f^1. Therefore, for all admissible sequences (words) including the periodic, eventually periodic, coarse-grained chaotic, and fined-grained chaotic sequences, under the joint actions of the operator $QC*$ and the mapping $f^{|QC|}$, the dynamical system can preserve the invariant of topological entropy, so the domain of second topological conjugate transformation is established in the whole symbolic space of two letters. It should be emphasized that, first, the infinitely many new equivalent classes of iterative maps ($\wp_{ss}*S$, $S\in\Sigma_2$, and $\wp_{ss}$ is the set of all superstable words) can be introduced by the second transformation, which are different from the infinitely many first equivalent classes ($QC*\Sigma_2$, $QC\in\wp_\pi$, and $\wp_\pi$ is the set of primitive words) in Ref. [3]. Second, they form a topological foundation of the renormalization-group operator if the joint actions of the operator $QC*$ and the mapping $f^{|QC|}$ repeat the same sequence an infinite number times; this also reflects the essence of the renormalization group from the aspect in which the right word and over-one-time mapping play the key roles. Third, the second topological conjugate transformation provides a topological method to discriminate compound words from all words in the order topological space Σ_2 of two letters; hence it may open up a way to seek the generalized $*$ product for a complex dynamical system of more than two letters [6]. Finally, it is interesting to define the inverse operation of the $*$ product on the basis of the second topological conjugate transformation; this may yield such ideas as fractional period and fractional renormalization group, which will be discussed in the future.

The paper is organized as follows. In Sec. II we discuss the ordering rule of the orbital points in the symbolic space of the compound word $QC*S$ and thus illustrate the block diagonalization of the Stefan transition matrix under mapping $f^{|QC|}$. In Sec. III we study the invariant of the second topological conjugate transformation. Section IV provides a topological method to discriminate compound words from all admissible words. In Sec. V we briefly illustrate Markov graphs under the second topological conjugate transformation. Finally, in Sec. VI we discuss the topological foundation for the renormalization-group operator and the infinitely many classes of topological conjugate iterative maps.

II. CLUSTERING OF ORDER IN SYMBOLIC SPACE

A. The π order of a compound word

The ordering rule of the orbital points in the symbolic space under the action of the operator $QC*$ is crucial for the whole paper. Before studying this we establish some basic notation of the symbolic dynamics. Consider a unimodal mapping [4] $f_\lambda: I\mapsto I$ over the interval $I=[-1,1]$ depending on a parameter λ. The location of the unique maximum of f_λ on the x axis can be normalized to be the origin of the kneading (maximal) sequence in the symbolic space. For an arbitrary point $x_0\in I$, the set $O_f(x_0)=\{f_\lambda^n(x_0)\}\equiv\{x_n\}$ denotes a trajectory with an initial point x_0. Each trajectory can be assigned an infinite sequence of three symbols L, C, and R,

$$W=w_1w_2\cdots,$$

where $w_i\in\{L,C,R\}$ is determined by the rule

$$w_i=\begin{cases} L & \text{if } x_i<0\\ C & \text{if } x_i=0\\ R & \text{if } x_i>0. \end{cases}$$

This sequence W is referred to as a word or an itinerary of the corresponding trajectory. All admissible words with arbitrary length s of symbols form a symbolic space Σ_2, namely, $\Sigma_2=\{W=\Pi_{i=1}^{s}w_i|s\in Z_+,w_i\in\{L,R\}\}$, or an order topological space Σ_2 if we assign the MSS order to each sequence W in the symbolic space. It is known that a superstable periodic word ends with a symbol C [4], denoted by $WC=w_1w_2\cdots w_pC$. In contrast, a nonsuperstable periodic trajectory with period p is repeated by the p-bit sequence $W=w_1w_2\cdots w_p$. They all are of course finite words and belong to Σ_2. Our discussion begins with finite words and then moves to infinite words.

The symbolic dynamics of a unimodal mapping $f_\lambda: I\mapsto I$ is described by the shift operator φ: $\Sigma_2\mapsto\Sigma_2$. Generally speaking, the mapping f_λ and shift φ are topologically conjugate [4,5]. Therefore, studying the dynamics of f_λ is equivalent to studying the dynamics of the conjugate shift φ with

$$\varphi(W)=w_2w_3\cdots.$$

The ordering of the orbital points $\{x_n\}$ in the coordinate space is naturally implied by the ordering of the real numbers. The ordering between words in the order topological space Σ_2 is just the MSS order $\succ$ [7,8]. Moreover, it not only corresponds to the ordering of real numbers in parameter space [9,10], but it also actually reflects the order in coordinate space of the mapping f_λ [11]. Therefore, we will study the MSS order that reflects the ordering between words in the coordinate space. In the following, we concentrate on the ordering of the word in the coordinate space.

For any two superstable words $WC=w_1w_2\cdots w_mC$, $m=|W|$, and $QC=q_1q_2\cdots q_nC$, $n=|Q|$, the DGP $*$ product [2] is defined as

$$QC*WC=q_1q_2\cdots q_nw_1^{t(Q)}q_1q_2\cdots q_nw_2^{t(Q)}q_1q_2\cdots q_n\cdots q_1q_2\cdots q_nw_m^{t(Q)}q_1q_2\cdots q_nC,$$

where t is the parity inverse operator defined by $L^t=R$, $R^t=L$, $t(Q)=t$ if Q contains an odd number of R's, and $t(Q)=I$ (identity operator) if Q contains an even number of R's. Now we establish a mapping π: $N\mapsto N$ by the MSS order such that the shift set $\varphi^0(WC),\varphi^1(WC),\ldots,\varphi^m(WC)$ is ordered as

$$\varphi^{\pi_{WC}(0)}(WC)>\varphi^{\pi_{WC}(1)}(WC)>\cdots>\varphi^{\pi_{WC}(m)}(WC),$$

where $\pi(0)=0$ because the word WC is always the kneading maximal word in Σ_2. It also can be shown that $\pi(m)=1$. $\{\pi_{WC}(i)\}_{i=0}^{i=|WC|-1}$ is called the π-order relation of the word WC [11].

Here we introduce two quantities $\overline{J}(S)$ and $\Phi_S(i)$. The R parity $\overline{J}(S)$ of a word S is defined as

$$\overline{J}(S)=\begin{cases}-1 & \text{if } S \text{ contains an odd number of } R\text{'s}\\ +1 & \text{if } S \text{ contains an even number of } R\text{'s}\end{cases}$$

and the keeping operator of the R parity is defined as

$$\Phi_S(i)=\frac{|\overline{J}(\varphi^{\pi_S(i)}(S))-\overline{J}(S)|}{2}=\begin{cases}0 & \text{if } \overline{J}(\varphi^{\pi_S(i)}(S))=\overline{J}(S), \quad \text{i.e., } R \text{ parity preserves}\\ 1 & \text{if } \overline{J}(\varphi^{\pi_S(i)}(S))\neq\overline{J}(S), \quad \text{i.e., } R \text{ parity inverts.}\end{cases}$$

Due to the order preservation of the operator $QC*$ [2,4], the following order relation for the compound word $QC*WC$ can be obtained:

$$QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC), \tag{1}$$

where

$$QC*\varphi^{\pi_{WC}(j)}(WC)=q_1q_2\cdots q_nw_{\pi_{WC}(j)+1}^{t(Q)}q_1q_2\cdots q_n\cdots q_1q_2\cdots q_nw_m^{t(Q)}q_1q_2\cdots q_nC.$$

In addition, we note the π-order relation of the superstable word QC,

$$\varphi^{\pi_{QC}(0)}(QC)>\varphi^{\pi_{QC}(1)}(QC)>\cdots>\varphi^{\pi_{QC}(n)}(QC). \tag{2}$$

After the shift operator $\varphi^{\pi_{QC}(i)}$, $i=0,1,...,n$, acts on the order relation (1) consecutively, we have

$$\begin{aligned}&\varphi^{\pi_{QC}(0)}(QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC))\\&>\varphi^{\pi_{QC}(1)}(QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC))\\&>\cdots>\varphi^{\pi_{QC}(n)}(QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC)).\end{aligned} \tag{3}$$

If $\varphi^{\pi_{QC}(i)}(QC)$ keeps the R parity of QC, i.e., $\Phi_{QC}(i)=0$, then the order relations (1) after shifting $\varphi^{\pi_{QC}(i)}$ will be preserved:

$$\begin{aligned}&\varphi^{\pi_{QC}(i)}(QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC))\\&\Rightarrow\varphi^{\pi_{QC}(i)}(QC*WC)>\varphi^{\pi_{QC}(i)}(QC*\varphi^{\pi_{WC}(1)}(WC))>\cdots>\varphi^{\pi_{QC}(i)}(QC*\varphi^{\pi_{WC}(j)}(WC))\\&\quad>\cdots>\varphi^{\pi_{QC}(i)}(QC*\varphi^{\pi_{WC}(m)}(WC))\\&\Rightarrow\varphi^{\pi_{QC}(i)}(QC*WC)>\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(1)}(QC*WC)>\cdots>\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(j)}(QC*WC)\\&\quad>\cdots>\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(m)}(QC*WC);\end{aligned} \tag{4a}$$

if $\varphi^{\pi_{QC}(i)}(QC)$ inverts the R parity of QC, i.e., $\Phi_{QC}(i)=1$, the order relations (1) after shifting $\varphi^{\pi_{QC}(i)}$ will be completely inverted as

$$\begin{aligned}&\varphi^{\pi_{QC}(i)}(QC*WC>QC*\varphi^{\pi_{WC}(1)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(j)}(WC)>\cdots>QC*\varphi^{\pi_{WC}(m)}(WC))\\&\Rightarrow\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(m)}(QC*WC)>\cdots>\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(j)}(QC*WC)>\cdots\\&\quad>\varphi^{\pi_{QC}(i)+|QC|\cdot\pi_{WC}(1)}(QC*WC)>\varphi^{\pi_{QC}(i)}(QC*WC).\end{aligned} \tag{4b}$$

With the order relation (2) of the word QC, we obtain the order relation of $|QC||WC|$ orbital points of the compound superstable word $GC=QC*WC$:

$$\varphi^{\pi_{GC}(0)}(GC) \succ \cdots \succ \varphi^{\pi_{GC}(j)}(GC) \succ \cdots \succ \varphi^{\pi_{GC}(m)}(GC) \quad (i=0)$$
$$\succ \cdots \succ \varphi^{\pi_{GC}(i(m+1))}(GC) \succ \cdots \succ \varphi^{\pi_{GC}(i(m+1)+j)}(GC) \succ \cdots \succ \varphi^{\pi_{GC}(i(m+1)+m)}(GC) \quad (i=i)$$
$$\succ \cdots \succ \varphi^{\pi_{GC}(n(m+1))}(GC) \succ \cdots \succ \varphi^{\pi_{GC}(n(m+1)+j)}(GC) \succ \cdots \succ \varphi^{\pi_{GC}((n+1)(m+1)-1)}(GC) \quad (i=n).$$

Consequently, for the compound sequence $QC*WC$, its π-order relations of orbital points in the symbolic space are given by the formula

$$\pi_{QC*WC}(i(m+1)+j) = \pi_{QC}(i) + (n+1)\cdot \pi_{WC}(\Phi_{QC}(i)\cdot m + (-1)^{\Phi_{QC}(i)} j), \tag{5}$$

where $i=0,\ldots,n$ and $j=0,\ldots,m$. In an analogous argument, the π-order relation formula for a compound eventually periodic word $QC*CAB^{\infty}$ (here strings A and B consist of L and R) can be obtained as

$$\pi_{QC*CAB^{\infty}}(iL_e+j) = \pi_{QC}(i) + (n+1)\cdot \pi_{CAB^{\infty}} \times (\Phi_{QC}(i)\cdot(L_e-1) + (-1)^{\Phi_{QC}(i)}\cdot j),$$

where $L_e = |CAB|$, $i=0,\ldots,n$ and $j=0,\ldots,L_e-1$.

For other coarse-grained chaotic words with limited grammatical rules [3] such as the limit of Fibonacci sequences [12] and the intermittent chaotic sequences [13], the operator $QC*$ can result in a similar π-order relation in the symbolic space [3,11]. Let $F_0=A$, $F_i=B$, and $B \succ A$; then we can generate the Fibonacci sequences as

$$F_2 = B \oplus A, \quad F_3 = F_2 \oplus F_1, \ldots, \quad F_n = F_{n-1} \oplus F_{n-2}, \ldots,$$

where the addition $\oplus$ [14] is defined as $F_{n-1} \oplus F_{n-2} := F_{n-1} R^{t(F_{n-1})} F_{n-2}$. The π-order relation of the Fibonacci sequences F_n can be analogously obtained as

$$\pi_{QC*F_n}(iL_{F_n}+j) = \pi_{QC}(i) + (n+1)\cdot \pi_{F_n} \times (\Phi_{QC}(i)\cdot(L_{F_n}-1) + (-1)^{\Phi_{QC}(i)}\cdot j),$$

where the length of the Fibonacci sequences F_n is given by

$$L_{F_n} = \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]|A| + \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]|B|\right\}.$$

When $n \to \infty$, we obtain the limit of the Fibonacci sequences to be the coarse-grained chaos. For the intermittent chaos, the π-order relation is similar and will be demonstrated by an example later.

Summing up the above discussion, we can generalize the conclusion from periodic words to aperiodic words and chaotic words. The effect of the operator $QC*$ on any symbolic words of a dynamical system will result in an order relation structure similar to formula (5).

B. The rule of transition order of a compound word

The π-order relations (5) demonstrate the cluster structure of $|QC|\cdot|WC|$ orbital points of the compound word $GC = QC*WC$. Shifting $\varphi^{|QC|}$ on the corresponding symbolic representations of orbital points in symbolic space means the periodic cycling of π order,

$$\varphi^{|QC|}: \ \pi_{QC*WC}(i(m+1)+j) \mapsto \pi_{QC*WC}(i(m+1)+j) + (n+1) \quad [\mathrm{mod}(n+1)(m+1)].$$

First, if the ith cluster keeps the R parity, i.e., $\Phi_{QC}(i)=0$, the π-order relation is given as

$$\pi_{QC*WC}(i(m+1)+j) \mapsto \pi_{QC}(i) + (n+1)\cdot \pi_{WC}(j)$$

and for this reason the mapping $\varphi^{|QC|}$ turns into

$$\varphi^{|QC|}: \ \pi_{QC}(i) + (n+1)\cdot \pi_{WC}(j) \mapsto \pi_{QC}(i) + (n+1)\cdot(\pi_{WC}(j)+1) \quad [\mathrm{mod}(n+1)(m+1)]. \tag{6}$$

It is obvious that the transition order of the iterative mapping $\varphi^{|QC|}$ is completely the same as the transition order under the mapping in the symbolic space of the word WC,

$$\varphi^1: \quad \pi_{WC}(j) \mapsto \pi_{WC}(j)+1 \quad [\mathrm{mod}(m+1)]. \tag{7}$$

Second, if the ith cluster inverts the R parity, i.e., $\Phi_{QC}(i) = 1$, the π-order relation would be

$$\pi_{QC*WC}(i(m+1)+j) = \pi_{QC}(i) + (n+1)\cdot \pi_{WC}(m-j).$$

It is certain that the transition order of the cluster under $\varphi^{|QC|}$ will be completely the inverse of Eq. (7). Therefore, it has been found that there are $|QC|$ new clusters appearing in symbolic space, each of them forming a self-closed subinterval, and the transition pattern under the mapping $f^{|QC|}$ in each cluster agrees with or inverts the original one of mapping f^1 in symbolic space.

On the basis of the above order clustering, the central results are introduced. The transition matrix (i.e., the Stefan matrix) of $|QC|\cdot|WC|$ orbital points of the compound word $QC*WC$ under the mapping $f^{|QC|}$ possesses the structure of $|QC|$ diagonal blocks and each block is a submatrix that would be either the same or the transpose of the transition matrix of mapping f^1 on $|WC|$ orbital points of the symbolic space. We will consider the block diagonalization of the Stefan matrix in detail in the next section.

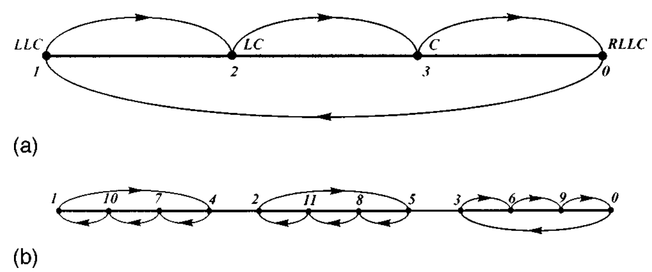

FIG. 1. (a) Transition pattern of mapping f^1 on the orbital points of the superstable word $RLLC$. (b) Transition pattern of mapping f^3 on the orbital points of the compound superstable word $RLC*RLLC$, on which a conspicuously clustering of the symbolic space exists.

III. THE SECOND TOPOLOGICAL CONJUGATE TRANSFORMATION

A. Topological entropy preservation

It has just been shown in the preceding section that the transformation of the orbital points yields the striking clustering in the symbolic space of the dynamical system and thus leads to the block diagonalization of its Stefan matrix, although there is a *connection line* between two adjacent diagonal blocks [such a line is marked, e.g., by the italic *1* in the matrix (9b)]. It can be easily shown by the simple property of the determinant from the matrix theory that these connection lines have no contribution to the eigenvalue of Stefan matrix and so the eigenvalue is determined by $|QC|$ blocks that have the same eigenvalue as the original Stefan matrix. We will show examples of the zero contribution of the connection line later. For one-dimensional unimodal maps, the value of the topological entropy can be calculated from the largest eigenvalue of the Stefan matrix [2]. We can immediately draw the conclusion that the second topological conjugate transformation preserves topological entropy, namely,

$$h_t(f^{|QC|}_{\lambda_{QC*S}}(x)) = h_t(f^1_{\lambda_S}(x)), \tag{8}$$

where the sequence S can be extended to all admissible sequences from finite words, such as the superstable words WC and their nonsuperstable window words, to infinite words, such as the eventually periodic word CAB^∞, limits of Fibonacci sequences, intermittent chaotic sequences, or any other chaotic sequences of dynamical systems when the lengths of words increase based on limited grammatical rules (or no rules) and approach infinity. Because the joint actions of the operator $QC*$ in parameter space and the mapping $f^{|QC|}$ on the symbolic interval do not change the topological property of dynamical systems in an infinite limit, the extension is valid. Here we give some examples to illustrate the transformation (8).

Example 1: The periodic words $QC = RLC$ *and* $WC = RLLC$. The π-order relation of the word $RLLC$ is given as (1,2,3,0) and its transition pattern under $f^1_{\lambda_{RLLC}}$ is shown in Fig. 1(a). The π-order relation of the word RLC is (1,2,0), i.e., $\pi_{QC}(0)=0$, $\pi_{QC}(1)=2$, $\pi_{QC}(2)=1$. After ordering $|QC|\cdot|WC| = 12$ orbital points of the word $QC*WC = RLLRLRRLRRLC$, we get its π-order relation (1,10,7,4|2,11,8,5|3,6,9,0). It obviously shows the order relation by formula (5), whereas the transition pattern of mapping $f^{|QC|}_{\lambda_{RLC*RLLC}} = f^3_{\lambda_{RLC*RLLC}}$ in Fig. 1(b) illustrates the clustering of symbolic space of the compound word $RLC*RLLC$.

Because $\varphi^{\pi_{QC}(0)}(RLC) = RLC$, $\varphi^{\pi_{QC}(1)}(RLC) = C$, $\varphi^{\pi_{QC}(2)}(RLC) = LC$, and thus $\Phi_{QC}(0) = 0$ and $\Phi_{QC}(1) = \Phi_{QC}(2) = 1$, the first cluster (i.e., the zeroth cluster) keeps the R parity and the transition pattern under f^3 in the cluster remains the same as that under f^1 in Fig. 1(a), while the other two clusters invert the R parity and the transition pattern under f^3 in each of these two clusters completely inverts to that under f^1 in Fig. 1(a). The corresponding Stefan matrices for the transition patterns in Figs. 1(a) and 1(b) are, respectively,

$$\mathsf{S}(f^1_{\lambda_{RLLC}}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{9a}$$

and

$$\mathsf{S}(f^3_{\lambda_{RLC*RLLC}}) = \begin{bmatrix}
1 & 1 & 1 & & & & & & & & \\
1 & 0 & 0 & & & & & & & & \\
0 & 1 & 0 & & & & & & & & \\
 & & \mathit{1} & \mathit{1} & \mathit{1} & \mathit{1} & \mathit{1} & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & & 1 & 0 & 0 & & & & \\
 & & & & 0 & 1 & 0 & & & & \\
 & & & & & & \mathit{1} & \mathit{1} & \mathit{1} & & \\
 & & & & & & & & 0 & 1 & 0 \\
 & & & & & & & & 0 & 0 & 1 \\
 & & & & & & & & 1 & 1 & 1
\end{bmatrix}. \tag{9b}$$

It is clear that the Stefan matrix of the transition pattern in Fig. 1(b) is diagonal and in its diagonal direction there exist three blocks that would be the same as or the transpose of the Stefan matrix of the transition pattern in Fig. 1(a). Although there are some other nonzero elements that form a connection line between every two adjacent blocks in the diagonal of matrix (9b), they contribute nothing to the eigenvalue of the matrix (9b); therefore, both Stefan matrices (9a) and (9b) have the same topological entropy and $h_t = (f^1_{\lambda_{RLLC}}) = h_t(f^3_{\lambda_{RLC*RLLC}}) = 0.609\,377\,863\,436\,0$.

Example 2: The eventually periodic word $CAB^\infty = CRLR^\infty$ *and* $QC = RLC$. The π-order relation of the eventually periodic word $CRLR^\infty$ is (2,0,3,1) and its transition pattern under f^1 is shown in Fig. 2(a). By ordering all orbital points of the compound eventually periodic word $QC*CAB^\infty = RLCRLLRLR(RLL)^\infty$, we obtain its π-order relation (4,10,1,7|5,11,2,8|6,0,9,3). Noticing the difference between periodic states and transient states, we have the two separate mapping formulas

$$\varphi^{|QC|}:\ \ \pi_{QC*CAB^\infty}(i)\mapsto\pi_{QC*CAB^\infty}(i)+|QC|$$

$$\text{if}\ \ \pi_{QC*CAB^\infty}(i)<|QC|\cdot|CA|,\ \text{ i.e., transient states turn to other states,}$$

$$\varphi^{|QC|}:\ \ \pi_{QC*CAB^\infty}(i)\mapsto|QC|\cdot|CA|+\{\pi_{QC*CAB^\infty}(i)-|QC|\cdot|CA|+|QC|\quad[\mathrm{mod}(|QC|\cdot|B|)]\}$$

$$\text{if}\ \ \pi_{QC*CAB^\infty}(i)\geqslant|QC|\cdot|CA|,\ \text{ i.e., periodic states turn to periodic states.}$$

The transition pattern under mapping f^3 in the symbolic space of the eventually compound periodic word $QC*CAB^\infty$ is shown in the Fig. 2(b) and it is clearly clustering in the symbolic space. The corresponding Stefan matrices for the transition patterns in Figs. 2(a) and 2(b) are, respectively,

$$\mathsf{S}(f^1_{\lambda_{CRLR^\infty}})=\begin{bmatrix}0&0&1\\0&0&1\\1&1&0\end{bmatrix}$$

and

$$\mathsf{S}(f^3_{\lambda_{RLC*CRLR^\infty}})=\begin{bmatrix}0&1&1&&&&&&&&\\1&0&0&&&&&&&&\\1&0&0&&&&&&&&\\&\mathit{1}&\mathit{1}&\mathit{1}&\mathit{1}&\mathit{1}&\mathit{1}&&&&\\&&&&0&1&1&&&&\\&&&&1&0&0&&&&\\&&&&1&0&0&&&&\\&&&&&\mathit{1}&\mathit{1}&\mathit{1}&\mathit{1}&\mathit{1}&\\&&&&&&&&0&0&1\\&&&&&&&&0&0&1\\&&&&&&&&1&1&0\end{bmatrix}.$$

These examples illustrate the clustering of the order relation of periodic and aperiodic words in the symbolic space and that of the corresponding block diagonalization of their Stefan matrices under the second topological conjugate transformation. Hence the topological entropy is guaranteed to be invariant and $h_t(f^1_{\lambda_{CRLR^\infty}})=h_t(f^3_{\lambda_{RLC*CRLR^\infty}})=0.346\,573\,590\,279\,972$.

In fact, in terms of the kneading theory by Milnor and Thurston [5], the invariant formula (8) can also be manifested. Take an admissible sequence $S=s_1s_2s_3\ldots$, where $s_j\in\{L,C,R\}$. One assigns the parity -1 to each R and $+1$ to each L. If the letter C appears, it is assigned the product of the parities of all the preceding letters. Let ε_j denote the parity of letter s_j; then the kneading determinant is defined by

$$P_{S,f^1}(\tau)=\sum_{n=0}^{\infty}\Theta^n\tau^n,\quad \Theta^n=\prod_{j=1}^{n}\varepsilon_j.$$

According to the $*$ product rule [2,11], the compound word $QC*S$ follows as

$$QC*S=(Qs_1^{t(Q)})(Qs_2^{t(Q)})(Qs_3^{t(Q)})\cdots.$$

Because the $DGP*$ product preserves the R parity [2,4], i.e., the parity of the block $Qs_j^{t(Q)}$ equals that of s_j, $\underline{J}(Qs_j^{t(Q)})=\underline{J}(s_j)$, if we start to iterate from the orbital point whose symbolic sequence is

$$q_i\cdots q_{|Q|}s_1^{t(Q)}q_1\cdots q_{|Q|}s_2^{t(Q)}q_1\cdots q_{|Q|}s_3^{t(Q)}\cdots,$$

$$i=1,\ldots,|Q|,$$

then the parity θ_j of the mapping

$$\begin{aligned}\varphi^{|QC|}:\ \ &q_i\cdots q_{|Q|}s_j^{t(Q)}q_1\cdots q_{|Q|}s_{j+1}^{t(Q)}\cdots\\&\mapsto q_i\cdots q_{|Q|}s_{j+1}^{t(Q)}q_1\cdots q_{|Q|}s_{j+2}^{t(Q)}\cdots,\\&\qquad i=1,\ldots,|Q|,\ \ j=1,\ldots,|S|,\end{aligned}$$

in the symbolic space of the compound word $QC*S$ is the same as that of φ^1 in the symbolic space of the word S, $\theta_j\equiv\theta(Qs_j^{t(Q)})=\varepsilon(s_j)$; therefore, the kneading determinant of $\varphi^{|QC|}$ on the compound word $QC*S$,

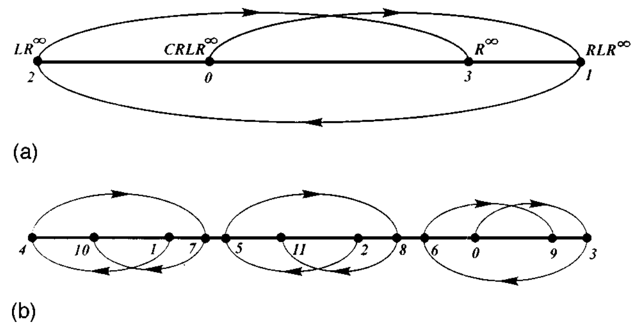

FIG. 2. (a) Transition pattern of mapping f^1 on the orbital points of the eventually periodic word $CRLR^\infty$. (b) Transition pattern of mapping f^3 on the orbital points of the compound eventually periodic word $RLC*CRLR^\infty$, on which a conspicuously clustering of the symbolic space exists.

$$P_{QC*S,f^{|QC|}}(\tau)=1+\sum_{n=1}^{\infty}\left(\prod_{j=1}^{n}\theta_j\right)\tau^n$$
$$=1+\sum_{n=1}^{\infty}\left(\prod_{j=1}^{n}\varepsilon_j\right)\tau^n=P_{S,f^1}(\tau).$$

As the topological entropy h_t is determined by the smallest positive root $\tau_{\min}$ of this characteristic polynomial $h_t=-\ln\tau_{\min}$ [2,8], we thus obtain formula (8).

It is worth noting that there are no restrictions to the sequence S in the procedure of the above proof; thus formula (8) is valid for all admissible sequences such as periodic, aperiodic, coarse-grained chaotic, or fine-grained chaotic sequences; namely, the second topological conjugate transformation preserves the topological entropy for any sequences $S\in\Sigma_2$.

Before ending this subsection we briefly clarify the coarse-grained chaos [10] in the frame of type CAB^∞ ($B\in\Sigma_2$). The coarse-grained chaos refers to the orbits with a positive Lyapunov exponent and a finite number of grammatical rules [3]. The structure of the coarse-grained chaotic orbits in phase space is complex, but most of them can be described by the symbolic type CAB^∞. They contain many important orbits such as the homoclinic points [10], band-merging points [10], crisis points [15], and Misiurewicz points [16,17]. When the stable set degenerates into a finite string A and then goes to the unstable set B^∞, it forms the homoclinic orbit, for example, $RL(RR)^\infty$. The merging points of 2^n with 2^{n-1} chaotic bands have the form AB^∞, where A and B stand for $(RC)^{*n}$ with the replacement of C by L or R such that A is odd and B even. For example, the $2\to1$ merging point is $RL(RR)^\infty$, the $4\to2$ merging point is $RLRR(RLRL)^\infty$, etc. While the unstable orbits collide with the chaotic attractor abruptly, these crisis points also possess the common form $Q*RL^\infty$ ($Q\in\wp_\pi$ the set of primitive words). Finally, the Misiurewicz points with a preperiod $|A|$ and an eventual period $|B|$ also belong to the type CAB^∞. We can see that the conclusion of the eventually periodic word in example 2 would be valid for the coarse-grained chaos here.

B. Examples of chaotic words

We now give further examples on the chaotic orbits to show the power of the previous theoretical results.

Example 3: Intermittent chaotic sequence. We consider such an intermittent chaotic sequence $(RLR)^\infty$ that appears just before period 3. The corresponding sequence can be written as [8]

$$\Lambda_k=R[(LRR)^kRR]^\infty,\quad k=1,2,\ldots\ .$$

It is obvious that

$$\Lambda_k\prec(LRR)^\infty\quad\forall k\geqslant1.$$

For sufficiently large k, these orbits spend most of the time traveling around as a period-3 pattern; the symbolic sequence is precisely the so-called intermittency. The results of the second topological conjugate transformation of the intermittent sequences Λ_k, $k=8,12,16,20$, are listed in Table I. We can see that the second topological conjugate transformation on the intermittent chaotic words preserves topological entropy.

Example 4: Limit of Fibonacci sequence. Selecting initial sequences $F_0=RC$ and $F_1=RLC$ and using the addition operation $\oplus$ in the symbolic space Σ_2 introduced by Peng [14], the Fibonacci sequences can be formed as

$$F_n=F_{n-1}\oplus F_{n-2};$$

for example,

$$F_2=RLC\oplus RC\equiv RLR^{t(RL)}RC=RLLRC,$$

$$F_3=RLLRRRLC,$$

$$F_4=RLLRRRLRRLLRC,$$

$$F_5=RLLRRRLRRLLRLRLLRRRLC.$$

The limit of the series F_n constructed in this way is a coarse-grained chaotic word. By a numerical calculation, it is impossible to reach its infinite limit. In order to investigate the character of the real limit, we could research the finite sequences and focus on what would happened under the sec-

TABLE I. $\Lambda_k=R[(LRR)^kRR]^\infty$. [Note that $h_t(f^1_{\lambda_{\Lambda_k}})<h_t(f^1_{\lambda_{(RLR)^\infty}})$.]

k	$h_t(f^1_{\lambda_{\Lambda_k}})$	$h_t(f^{\lvert RC\rvert}_{\lambda_{RC*\Lambda_k}})$	$h_t(f^{\lvert RLC\rvert}_{\lambda_{RLC*\Lambda_k}})$
8	0.481 209 788 480 959	0.481 209 788 480 959	0.481 209 788 480 959
12	0.481 211 818 735 089	0.481 211 818 735 089	0.481 211 818 735 089
16	0.481 211 825 039 962	0.481 211 825 039 962	0.481 211 825 039 962
20	0.481 211 825 059 542	0.481 211 825 059 542	0.481 211 825 059 542

TABLE II. $F_n = F_{n-1} \oplus F_{n-2}$ with $F_0 = RC$ and $F_1 = RLC$.

k	$h_t(f^1_{\lambda_{F_n}})$	$h_t(f^{\lvert RC\rvert}_{\lambda_{RC*F_n}})$	$h_t(f^{\lvert RLC\rvert}_{\lambda_{RLC*F_n}})$
6	0.547 665 729 752 595	0.547 665 729 752 595	0.547 665 729 752 595
7	0.547 665 773 919 357	0.547 665 773 919 357	0.547 665 773 919 357
8	0.547 665 773 919 804	0.547 665 773 919 804	0.547 665 773 919 804
9	0.547 665 773 919 804	0.547 665 773 919 804	0.547 665 773 919 804

ond topological conjugate transformation. The results for sequences F_n, $n = 6-9$, are listed in Table II. We assert from the results of the finite step sequences that the topological entropies of the limit of Fibonacci sequences are preserved under the second topological conjugate transformation.

C. Topological entropy of iterative maps and of the second topological conjugate transformation

According to the first topological conjugate transformation [3], there exists the following general relation for topological entropy under the action of the operator $QC*$:

$$h_t(f^1_{\lambda_{QC*S}}) = \begin{cases} h_t(f^1_{\lambda_{QC}}) & \text{if } QC \neq (RC)^{*n}, \quad n \in Z_+ \\ h_t(f^1_{\lambda_S})/2^n & \text{if } QC = (RC)^{*n}, \quad n \in Z_+. \end{cases} \tag{10}$$

However, it is well known from the ergodic theory [18] that there exists a formula for calculating the topological entropy of k times iterations of the map f:

$$h_t(f^k_\lambda) = k \cdot h_t(f^1_\lambda), \tag{11}$$

where $k \in Z_+$. Thus, for the case $QC \neq (RC)^{*n}$, we have

$$h_t(f^k_{\lambda_{QC*S}}) = k \cdot h_t(f^1_{\lambda_{QC*S}}) = k \cdot h_t(f^1_{\lambda_{QC}}).$$

Therefore, it seems that there exists an apparent inconsistency between formulas (8) and (11). What mistake does this paradox result from? We will discuss the problem by analyzing the concrete examples and show how we should explain formula (11) in terms of symbolic dynamics.

Example 5: The periodic sequences $WC = RC$ and $QC = RLC$. The transition pattern and Stefan matrix under mapping f^1 on the orbital points of the compound word $QC*WC = RLLRLC$ are shown respectively in Fig. 3(a) and in the expression of the matrix

$$\mathsf{S}(f^1_{\lambda_{RLC*RC}}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding largest eigenvalue determined by the equation $(1-\lambda^3)(\lambda^2-\lambda-1)=0$ is given by $\lambda_m = (1+\sqrt{5})/2$. So $h_t(f^1_{\lambda_{RLC*RC}}) = h_t(f^1_{\lambda_{RLC}}) = 0.481\,211\,825\ldots$, which obeys formula (10). On the other hand, the transition relationship and Stefan matrix under the mapping $f^{|RLC|} = f^3$ on the same orbital points are shown in Fig. 3(b) and in the expression of the matrix

$$\mathsf{S}(f^3_{\lambda_{RLC*RC}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding largest eigenvalue determined by the equation $(1-\lambda)^5 = 0$ is $\lambda_m = 1$, so $h_t(f^3_{\lambda_{RLC*RC}}) = h_t(f^1_{\lambda_{RC}}) = 0$, which is consistent with formula (8). However, if the first preimages and the second preimages of all orbital points in the dynamical invariant interval are picked out first, then the mapping f^3 on the union of the original orbital points, the first preimages, and the second preimages leads to the transition relationship in Fig. 3(c) and the Stefan matrix is

$$\mathsf{S}(f^3_{\lambda_{RLC*RC}}\{O_f \cup O_{f^{-1}} \cup O_{f^{-2}}\}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

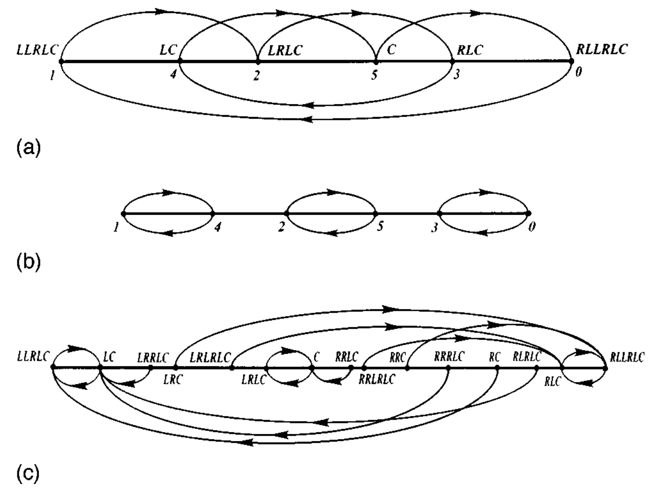

FIG. 3. (a) Transition pattern of mapping f^1 on the orbital points of the compound superstable word $RLC*RC$. (b) Transition pattern of mapping f^3 on the orbital points of the compound superstable word $RLC*RC$. (c) Transition pattern of mapping f^3 on the set of points $O_f \cup O_{f^{-1}} \cup O_{f^{-2}}$. Here O_f represents the set of orbital points, $O_{f^{-n}}$ the nth preimages of orbital points. For the superstable word $RLC*RC$, $O_f = \{LLRLC, LC, LRLC, C, RLC, RLLRLC\}$, $O_{f^{-1}} = \{RRLC, RC, RLRLC\}$, and $O_{f^{-2}} = \{LRRLC, LRC, LRLRLC, RRLRLC, RRC, RRRLC\}$.

The largest eigenvalue of the above Stefan matrix follows as $\lambda_m = \sqrt{5} + 2 = [(1+\sqrt{5})/2]^3$; therefore,

$$h_t(f^3_{\lambda_{RLC*RC}}\{O_f \cup O_{f^{-1}} \cup O_{f^{-2}}\}) = 3h_t(f^1_{\lambda_{RLC*RC}}\{O_f\}) = 3h_t(f^1_{\lambda_{RLC}}\{O_f\}),$$

which is consistent with formula (11). Here O_f represents the set of $|QC*WC|$ orbital points and $O_{f^{-1}}$ and $O_{f^{-2}}$ represent the sets of the first preimages and second preimages of all original orbital points in the dynamical invariant interval, $f^3_{\lambda_{RLC*RC}}\{O_f \cup O_{f^{-1}} \cup O_{f^{-2}}\}$ denotes the mapping f^3 for the parameter λ_{RLC*RC} on the set of points $O_f \cup O_{f^{-1}} \cup O_{f^{-2}}$.

Example 6: The eventually periodic sequence $CRLLR^\infty$. The transition pattern and Stefan matrix under the mapping f^1 on the point set O_f of $CRLLR^\infty$ are shown in Fig. 4(a) and its Stefan matrix is

$$S(f^1_{\lambda_{CRLLR^\infty}}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalue equation is $\lambda(\lambda^3 - 2\lambda - 2) = 0$. In contrast, the transition pattern and Stefan matrix under mapping f^2 on O_f is shown in Fig. 4(b) and in the expression

$$S(f^2_{\lambda_{CRLLR^\infty}}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

It is easy to see that all these are inappropriate and should be given up. With all the preimages of O_f in the dynamical invariant interval, the mapping f^1 on the set of points $O_f \cup O_{f^{-1}}$ leads to the transition pattern in Fig. 4(c) and its Stefan matrix is

$$S(f^1_{\lambda_{CRLLR^\infty}}\{O_f \cup O_{f^{-1}}\}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalue equation is still $\lambda^4(\lambda^3 - 2\lambda - 2) = 0$. As a result, the topological entropy is preserved in the first refinement partition of the symbolic space by the set of points $O_{f^{-1}}$ [19]. The mapping f^2 on the set of points $O_f \cup O_{f^{-1}}$ leads to the transition pattern in Fig. 4(d) and its Stefan matrix is

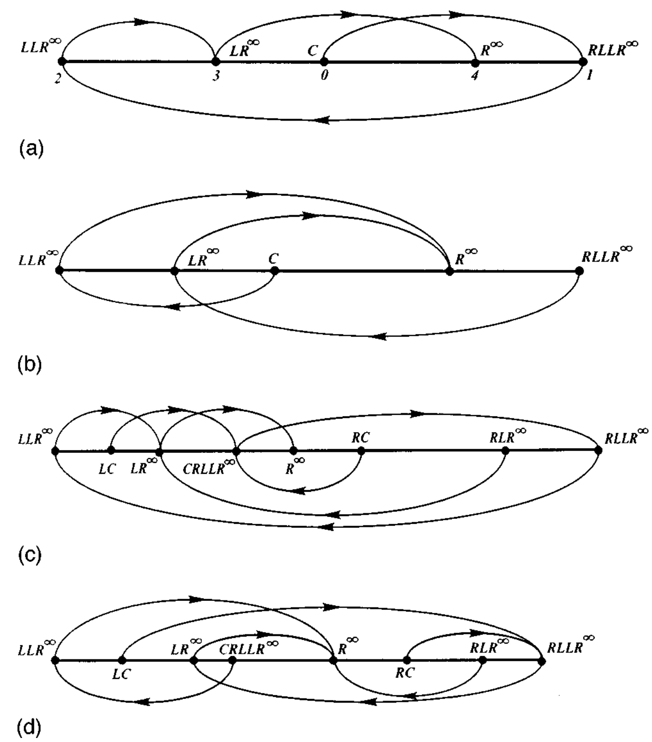

FIG. 4. (a) Transition pattern of mapping f^1 on the orbital points of the eventually periodic word $CRLLR^\infty$. $O_f = \{LLR^\infty, LR^\infty, CRLLR^\infty, R^\infty, RLLR^\infty\}$. (b) Transition pattern of mapping f^2 on the orbital points of the eventually periodic word $CRLLR^\infty$. Note that the subinterval $[LLR^\infty, LR^\infty]$ is mapped onto a single point, which is inappropriate. (c) Transition pattern of mapping f^1 on the set of points $O_f \cup O_{f^{-1}}$ of the eventually periodic word $CRLLR^\infty$; here $O_{f^{-1}} = \{LC, RC, RLR^\infty\}$. In this case the inappropriate iteration does not exist. (d) Transition pattern of mapping f^2 on the set of points $O_f \cup O_{f^{-1}}$ of the eventually periodic word $CRLLR^\infty$.

$$\mathsf{S}(f^2_{\lambda_{CRLLR^\infty}}\{O_f \cup O_{f^{-1}}\}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalue equation is

$$\lambda^4(\lambda^3 - 4\lambda^2 + 4\lambda - 4) = \lambda^4[(\sqrt{\lambda})^3 - 2\sqrt{\lambda} - 2] \times [(\sqrt{\lambda})^3 - 2\sqrt{\lambda} + 2] = 0.$$

Evidently, the largest eigenvalue is determined by the equation $[(\sqrt{\lambda})^3 - 2\sqrt{\lambda} - 2] = 0$, so we obtain the result

$$h_t(f^2_{\lambda_{CRLLR^\infty}}\{O_f \cup O_{f^{-1}}\}) = 2h_t(f^1_{\lambda_{CRLLR^\infty}}\{O_f\}) = 2h_t(f^1_{\lambda_{CRLLR^\infty}}\{O_f \cup O_{f^{-1}}\}).$$

This is what formula (11) means.

Summarizing the above two illustrations, we can explicitly rewrite formula (11) as

$$h_t(f^k_\lambda\{O_f \cup O_{f^{-1}} \cdots \cup O_{f^{-(k-1)}}\}) = k \cdot h_t(f^k_\lambda\{O_f\}),$$

where $O_{f^{-n}}$ represents the nth preimages of the set of orbital points O_f and formula (8) as

$$h_t(f^{|QC|}_{\lambda_{QC*S}}\{O_f\}) = h_t(f^1_{\lambda_S}\{O_f\}).$$

Then the specious inconsistency vanishes naturally. It has clarified the difference between the above two important formulas of topological entropy in the sense of the symbolic dynamics.

IV. A TOPOLOGICAL METHOD OF DISTINGUISHING COMPOUND WORDS

Given any superstable word QC and any admissible word S, we can easily construct a compound word $QC*S$. In the symbolic space Σ_2, there are many more compound words than primitive ones [3]. The inverse question arises naturally how to locate compound words or how to decide whether a word is primitive or compound.

The answer becomes evident when taking advantage of the second topological conjugate transformation. For any superstable word GC with the period being factorized as $|GC| = k_1 \cdot k_2$ $(k_1, k_2 \in Z_+)$, we inspect the transition pattern of the mappings f^{k_1}, f^{k_2} on its orbital points. If there are k_i $(i=1$ or $2)$ self-closed clusters under the mapping f^{k_i} in the symbolic space and the transition patterns on these clusters are the same or the inverse of each other, then the word can definitely be factorized in the sense of the DGP $*$ product. Further, by calculating the topological entropies of f^1 and f^{k_i} on the orbital points respectively, say $h_t(f^1, GC)$ and $h_t(f^{k_i}, GC)$, and using the properties of the devil's staircase of the topological entropy [3], the result follows that the word GC can be factorized as a word with topological entropy $h_t(f^1, GC)$ and period k_i multiplying by another word with $h_t(f^{k_i}, GC)$ and period $|GC|/k_i$ if $h_t(f^{k_i}, GC) \neq h_t(f^1, GC)/2^n$; if $h_t(f^{k_i}, GC) = h_t(f^1, GC)/2^n$ and $k_i = 2^n$, then the word GC is factorized as $(RC)^{*n} * S$ and the word S has topological entropy $h_t(f^{k_i}, GC)$ and period $|GC|/2^n$. Otherwise, GC is primitive. Since the DCP $*$ product is noncommutative, the factorization of a compound word exists uniquely.

Example 7: There are five superstable words with period 6, $RLRRRC$, $RLLRLC$, $RLLRRC$, $RLLLRC$, and $RLLLLC$. The transition patterns under the mappings f^2, f^3 on these words are respectively shown in Figs. 5(a) and 5(b). So only two words $RLRRRC$ and $RLLRLC$ are compound and the rest are primitive. Computing the topological entropies yields

$$h_t(f^1, RLRRRC) = h_t(f^2, RLRRRC)/2 = 0.240\,605\,912\,529\,802\ldots,$$

$$h_t(f^1, RLLRLC) = h_t(f^1, RLC) = 0.481\,211\,825\,059\,603\ldots,$$

$$h_t(f^3, RLLRLC) = h_t(f^1, RC) = 0.$$

We thus have the results that two words can be factorized respectively as

$$RLRRRC = RC*RLC, \quad RLLRLC = RLC*RC.$$

In general, if the period of a word GC can be factorized as $|GC| = k_1 k_2 \cdots k_l$ we watch carefully the transition patterns of the mappings

$$f^{k_1}, f^{k_2}, \ldots, f^{k_l}, f^{k_1 k_2}, \ldots, f^{k_1 k_l}, f^{k_2 k_3}, \ldots, f^{k_2 k_l}, \ldots, f^{k_{l-1} k_l}, \ldots, f^{k_1 k_2 k_3}, \ldots, f^{k_1 k_2 \cdots k_l}$$

on the orbital points of the word GC. Then by the above method we can judge whether or not the word GC can be factorized. Therefore, this new topological conjugate transformation opens up a way to discriminate compound words in the symbolic space Σ_2 of two letters and even in Σ_3 of three letters, etc. Further, it provides a method to seek the generalization of the DGP $*$ product in more complex dynamical systems; for instance, by this method the dual $*$ products in the symbolic space Σ_3 of three letters have been constructed [6].

In addition, on the basis of the second topological conjugate transformation we can introduce the inverse operation of the $*$ product in some sense, by which we have already shown that the method can be used to separate a primitive word from the compound words. Here we further conjecture that the inverse operation of the $*$ product may generalize the concept of noncompound words. We now attempt to extend the concept of period such that a word may have a fractional period [20]. Let WC be a superstable periodic word with period $|WC| = m$. If the transition pattern of compound mapping f^n on the orbital points of WC is appropriate and its topological entropy is calculated to be h^*, then we can define a word (state) with fractional period, namely, $f^n_{\lambda^*}(x) = f^m_{\lambda^*}(x)$, where m is prime to n, the parameter λ^* is fixed, and this word has a topological entropy value h^*. When changing the value of its parameter according to the $*$ prod-

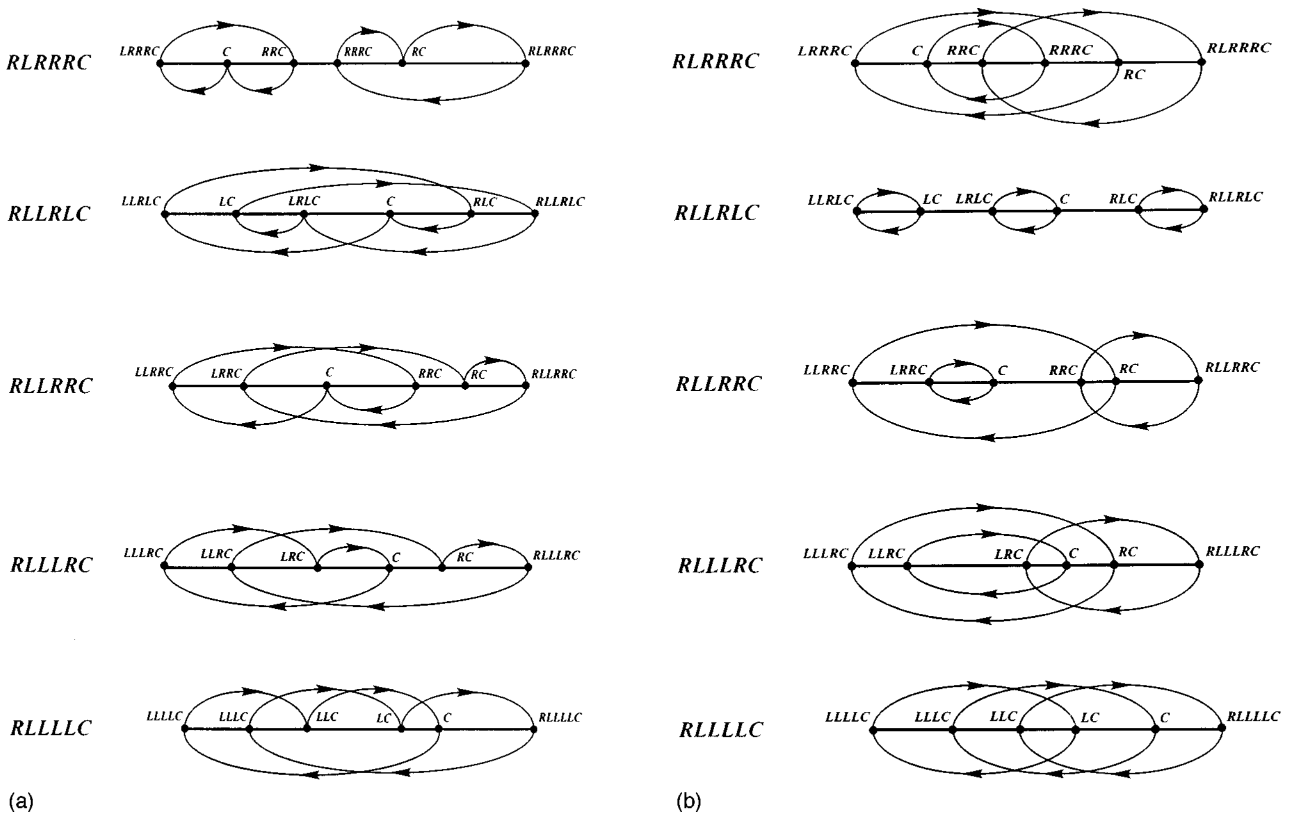

FIG. 5. (a) Transition pattern of mapping f^2 on the orbital points of superstable words $RLRRRC$, $RLLRLC$, $RLLRRC$, $RLLLRC$, and $RLLLLC$. Only in the symbolic space of the word $RLRRRC$ does clustering of the transition order exist. (b) Transition pattern of mapping f^3 on the orbital points of superstable words $RLRRRC$, $RLLRLC$, $RLLRRC$, $RLLLRC$, and $RLLLLC$. Only in the symbolic space of the word $RLLRLC$ does clustering of the transition order exist.

uct rule, we may use the inverse operation of the $*$ product to seek a new equal topological entropy class with a fractional period. If this is the case, there may exist a fractional renormalization-group equation such as $g^n(x) = \alpha g^m(x/\alpha)$ [20]. This may be an interesting type of dynamical word (state).

V. MARKOV GRAPHS UNDER THE SECOND TOPOLOGICAL CONJUGATE TRANSFORMATION

The Stefan matrices above are explained as the transitions of intervals. Moreover, it is very useful to explain them as the Markov graphs of the transitions of points (states). From the graph theory, the transitions of both intervals and points are equivalent in the sense of duality. However, we can understand much more information from Markov graph such as the symbolic kinetic analysis [21], the stationary measure of the topological Markov chain [22], and the theoretical analysis of formal language of finite automata from the periodic, eventually periodic, and aperiodic unimodal maps [23–25]. Here we restrict our discussion to the second topological conjugate transformation by the Markov graph. An important feature is that the probability measure of dynamical systems in the Markov graph will be expressed more clearly than in the graph of interval transitions. We choose the examples from above to compare these two kinds of graphs.

A. The periodic orbit

As we know, it is very easy to draw the Markov graph from the Stefan matrix of the finite periodic orbit. Thus the interval of ''mass'' points in the graph of interval transitions would be shrunk into a point that supplies or receives the transfer of mass points and these two kinds of points are called the source and sink, respectively. The Markov graph of the periodic orbit $RLRRRC$ of the map f^1 is drawn in Fig. 6(a) and that of the map f^2 in Fig. 6(c). We can see that these two Markov graphs have the same transition pattern, except the point $RRRR$, which is a source to supply the same two subgraphs with mass points in Fig. 6(c). The point $RRRR$ only as the source does not contribute to the topological entropy. In fact, only such points that are not only source but also sink may possibly contribute to the topological entropy.

B. The eventually periodic orbit

The eventually periodic orbits are the simplest example of coarse-grained chaos. They contain many important orbits such as the homoclinic points, crisis points, and band-merging points and have the common symbolic description CAB^∞. We adopt one of the typical examples discussed above (example 2) [Figs. 2(a) and 2(b)]. The Markov graphs of the eventually periodic orbit $CAB^\infty = CRLR^\infty$ and the compound word $RLC*CRLR^\infty = RLCRLLRLR(RLL)^\infty$ are drawn in Figs. 7(a) and 7(b). Of course, the conclusion is the same as the previous periodic orbit, namely, all three Markov subgraphs in Fig. 7(b) are the same as Fig. 7(a), except the two sources from $d=[7,5]$ and $h=[8,6]$.

In order to understand completely, we interpret this again with the analysis of symbolic kinematics [21]. For the eventually periodic orbit $CRLR^\infty$ of the unimodal map f^1 the rule

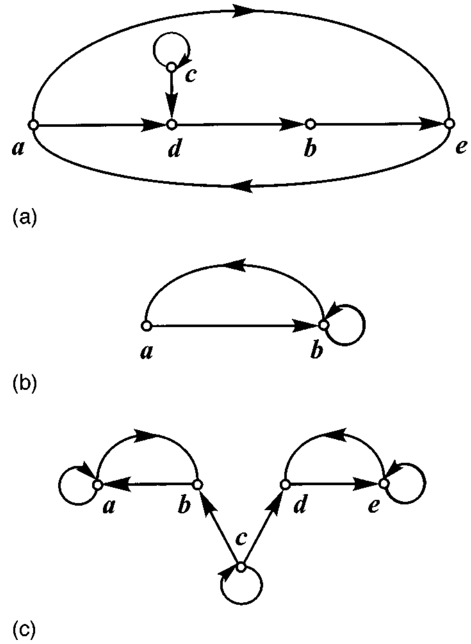

FIG. 6. (a) Markov graph of the compound superstable word $RC*RLC=RLRRRC$ of the map f^1. The intervals are denoted by $a=LR$, $b=RRLR$, $c=RRRR$, $d=RRRLR$, and $e=RLR$. (b) Markov graph of the periodic orbit RLC of the map f^1. The intervals are denoted by $a=LR$ and $b=R$. (c) Markov graph of the periodic orbit $RLRRRC$ of the map f^2, which is a series connecting the graph in (b). The intervals are the same as in (a).

of transition of orbital points in the Markov graph is given in Ref. [21]. Carrying out the transition one time between two intervals with different masses (namely, the interval lengths), we can calculate its transition probability by the interval lengths. If we suppose that the interval length of each transition point is equal to one, then the Stefan matrix can be interpreted as a topological Markov chain. If we take successively the inverse maps $f^{-1}, f^{-2}, \ldots, f^{-n}$, then the entropy of

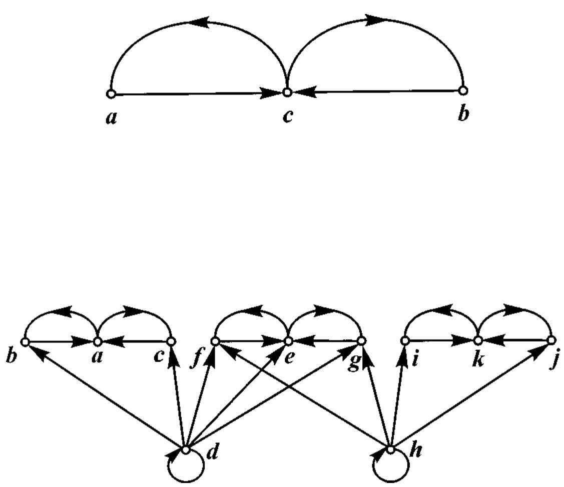

FIG. 7. (a) Markov graph of the eventually periodic word $CRLR^\infty$ of the map f^1. The intervals are denoted by $a=[2,0]$, $b=[0,3]$, and $c=[3,1]$ according to Fig. 2(a). (b) Markov graph of the eventually periodic word $RLC*CRLR^\infty$ of the map f^3. The intervals are denoted by $a=[4,10]$, $b=[10,1]$, $c=[1,7]$, $d=[7,5]$, $e=[5,11]$, $f=[11,2]$, $g=[2,8]$, $h=[8,6]$, $i=[6,0]$, $j=[0,9]$, and $k=[9,3]$ according to Fig. 2(b).

the topological Markov chain of the one-sided shift can be easily obtained as [22] $m=-\Sigma_{i,j} p_j P_{ij} \ln P_{ij}$, where the invariant probability $p_j=\Sigma_i p_i P_{ij}$, the transition probabilities $P_{ij}=s_{ij} z_j / z_i \lambda(\mathrm{S})$, $\lambda(\mathrm{S})$ is the maximal positive eigenvalue of the Stefan matrix of f^{-n}, and $Z=\{z_i\}$ is the corresponding eigenvector. Moreover, other coarse-grained chaotic sequences discussed in Sec. III B would have the same conclusion in the Markov graphs when we approach the limit of an infinite sequence from the finite one.

Summing up the analysis of Markov graphs in this section, we have that under the second topological conjugate transformation, the $|QC|$ subgraphs of the map $f^{|QC|}_{\lambda_{QC*S}}$ of a compound word $QC*S$ are the same as the graph of the map f_{λ_S} of the word S except for the sources, which do not contribute to the topological entropy, and both $f^{|QC|}_{\lambda_{QC*S}}$ and f_{λ_S} have the same probability measure of the transition with maximal entropy for the topological Markov chain.

Finally, we end this section with a brief discussion about automata. If the finite automaton S is deduced by the Stefan matrix of the symbolic sequence SC, then the Stefan matrix of the symbolic sequence $QC*S$ under the second topological conjugate transformation will generate the finite automaton S', which is in a series of $|QC|$ finite automata S connected by $|Q|$ sources. Both S and S' are equivalent in the grammatical rule of formal language theory [26]. This is also one way of constructing the automata with equal topological entropy.

VI. TOPOLOGICAL CONJUGATE CLASS OF ITERATIVE MAPS

A. Infinite number of topological conjugate classes of iterative maps

We already know that the invariant of the topological entropy by the $*$ product $h_t(f_{\lambda_{QC*W}})=h_t(f_{\lambda_{QC}})$ leads to the first topological conjugate class, which is labeled by each primitive word $QC \in \wp_\pi$ (the set of primitive words), namely, a step of the devil's staircase of topological entropy on the parametric axis [3]. Is it possible to establish another topological conjugate relationship between mappings f^1 and f^n, $n \in Z_+$? The answer is yes. Analogously, the invariant of the second topological conjugate relationship $h_t(f^{|QC|}_{\lambda_{QC*S}}\{O_f\})=h_t(f^1_{\lambda_S}\{O_f\})$ can also give rise to a topological conjugate class between the mappings f^1 and f^n. Given the mapping $f^1_{\lambda_S}$ with the parameter λ_S represented by an arbitrary admissible sequence S in the unimodal map, we can choose the appropriate parameter λ_{QC*S} with $|QC|=n$. According to the second topological conjugate transformation, the mappings $f^1_{\lambda_S}$ and $f^n_{\lambda_{QC*S}}$ are topologically conjugate in the symbolic space. Therefore, in terms of topology, the physical orbit of S in the symbolic space under $f^1_{\lambda_S}$ is topologically the same as that on each cluster of $QC*S$ in the symbolic space under $f^n_{\lambda_{QC*S}}$. When the admissible sequence S is fixed, which definitely belongs to one of the steps of the devil's staircase of the topological entropy on the parametric axis [3,27] (say, $S \in PC*[L^\infty, RL^\infty]$) or one of the single points of chaos, if the left word QC runs over the set $\wp_{ss}$ of all admissible superstable words (namely, QC takes from all the infinitely many steps of the entropy staircase), then a second topological conjugate class is formed for a concrete S. Because the set of the first topological conjugate classes has cardinal number $\aleph_1$, a second topological conjugate class also has cardinal number $\aleph_1$. In addition, in view of the invariance of the first topological conjugate class, the admissible sequence S can take over sequences from (a) one of the steps of the entropy staircase that contains infinitely many admissible sequences and corresponds to a first topological conjugate class with cardinal number $\aleph_1$ or (b) the infinitely many steps of the entropy staircase or the infinitely many single points of chaos. Thus all the infinitely many second topological conjugate classes form a set. It is obvious that the cardinal number of the set of the second topological conjugate classes should also be the same as that of the set of the first topological conjugate classes (namely, $\aleph_1$).

Therefore, it is clear that these two kinds of topological conjugate classes reflect an entropic invariant of the symbolic dynamics in two different ways. As far as the invariance of topological entropy is concerned, the motion of the dynamical systems is measured with the same criterion f^1 in the first topological conjugate class, while the motion of different dynamical systems in the second topological conjugate class is measured with different criteria $f^{|QC|}$. However, the second equivalent class has a noticeable effect on exploring the metric characteristics. The method of measuring different objects by different criteria and thus seeking the invariant in between is effective for studying characteristics or features of chaos such as the self-similarity. It becomes evident in the following that the second topological conjugate transformation supplies the topological foundation for the renormalization-group operator.

B. Topological foundation of the renormalization-group operator

It is well known that the renormalization-group operator in the Feigenbaum bifurcation processes $(QC)^{*n}$, $n=1,2,\ldots,$ is [1,8]

$$Tf_{\lambda_{(QC)^{*n}}}(x)=\alpha f^{|QC|}_{\lambda_{QC*(QC)^{*n}}}(x/\alpha).$$

In terms of topology, this operator is equivalent to the second topological conjugate transformation in the symbolic space at the parameter $\lambda_{(QC)^{*n}}$. Owing to formula (8), it leads to

$$h_t(f_{\lambda_{QC}})=h_t(f^{|QC|}_{\lambda_{QC*QC}})=\cdots=h_t(f^{|(QC)^{*n}|}_{\lambda_{(QC)^{*n}*QC}})$$

and in particular for the accumulation points $\lambda_\infty=\lambda_{(QC)^{*\infty}}$.

$$h_t(g^{|QC|}_{\lambda_{QC*(QC)^{*\infty}}})=h_t(g^1_{\lambda_{(QC)^{*\infty}}}),$$

where $g(x)=g_{\lambda_\infty}(x)=f^{|(QC)^{*\infty}|}_{\lambda_{(QC)^{*\infty}}}(x)$ are the fixed-point functions of the renormalization operator T. That is, the second topological conjugate transformation guarantees that the Feigenbaum bifurcation processes preserve topological entropy [3]. The fixed-point functions $g(x)$ of the operator T at the accumulation points λ_∞, are also the fixed-point functions of the second topological conjugate transformation and g^1 and $g^{|QC|}$ are conjugate at the same accumulation points λ_∞. Thus the second topological conjugate transformation supplies a topological foundation of Feigenbaum's universalities and a sharp topological frame in studying the transitions to chaos in Feigenbaum's scenario.

Knowledge of the topological characteristics would further the exploration of the metric regularities [27] of the dynamical systems. The second topological conjugate transformation will reveal some new regularities for the invariant distribution of the phase space and for the metric entropy, in particular for the Lyapunov exponent [28]. We believe that this topological framework for the research of the metric characteristics in the chaotic dynamical systems would be significant in the future.

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